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STUDY PACKAGE

Subject : Mathematics
Topic : INVERSE TRIGONOMETRY

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Inverse Circular Functions

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function		Domain	Range
(i)	$y = \sin^{-1}x$	where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	where	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \text{cosec}^{-1}x$	where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(v)	$y = \sec^{-1}x$	where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi;\ y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1}x$	where	$x \in R$	$0 < y < \pi$

NOTE:

(a) 1st quadrant is common to the range of all the inverse functions.

(b) 3rd quadrant is not used in inverse functions.

(c) 4th quadrant is used in the clockwise direction i.e. $-\frac{\pi}{2} \le y \le 0$.

(d) No inverse function is periodic. **(See the graphs on page 17)**

Solved Example # 1

Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Solution

Let $y = \tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

$\because$ $= \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right]$

$= \tan\left(\frac{\pi}{6}\right)$

$y = \frac{1}{\sqrt{3}}$ **Ans.**

Self practice problems:

Find the value of the followings :

(1) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ **Ans.** 1

(2) $\text{cosec}\,[\sec^{-1}(-\sqrt{2}) + \cot^{-1}(-1)]$ **Ans.** -1

Solved Example # 2

Find domain of $\sin^{-1}(2x^2 - 1)$

Solution.

Let $y = \sin^{-1}(2x^2 - 1)$

For y to be defined

$-1 \le (2x^2 - 1) \le 1$

$\Rightarrow$ $0 \le 2x^2 \le 2$

$\Rightarrow$ $0 \le x^2 \le 1$

$\Rightarrow$ $\mathbf{x \in [-1, 1]}$

Self practice problems:

Find the domain of followings :

(3) $y = \sec^{-1}(x^2 + 3x + 1)$

(4) $y = \cos^{-1}\left(\dfrac{x^2}{1+x^2}\right)$

(5) $y = \tan^{-1}(\sqrt{x^2-1})$

Answers (3) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$
(4) R
(5) $(-\infty, -1] \cup [1, \infty)$

2. Properties of Inverse Trigonometric Functions:

Property - 2(A)

(i) $\sin(\sin^{-1}x) = x, \quad -1 \le x \le 1$ (ii) $\cos(\cos^{-1}x) = x, \quad -1 \le x \le 1$

(iii) $\tan(\tan^{-1}x) = x, \quad x \in R$ (iv) $\cot(\cot^{-1}x) = x, \quad x \in R$

(v) $\sec(\sec^{-1}x) = x, \quad x \le -1, x \ge 1$ (vi) $\text{cosec}(\text{cosec}^{-1}x) = x, \quad x \le -1, x \ge 1$

These functions are equal to identity function in their whole domain which may or may not be R. **(See the graphs on page 18)**

Solved Example # 3

Find the value of cosec $\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\}$.

Solution.

Let $y = \text{cosec}\left\{\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right)\right\}$ ....(i)

$\because$ **$\cot(\cot^{-1}x) = x, \ \forall x \in R$**

$\therefore$ $\cot\left(\cot^{-1}\dfrac{3\pi}{4}\right) = \dfrac{3\pi}{4}$

$\therefore$ from equation (i), we get

$y = \text{cosec}\left(\dfrac{3\pi}{4}\right)$

$y = \sqrt{2}$ Ans.

Self practice problems:

Find the value of each of the following :

(6) $\cos\left\{\sin\left(\sin^{-1}\dfrac{\pi}{6}\right)\right\}$ (7) $\sin\left\{\cos\left(\cos^{-1}\dfrac{3\pi}{4}\right)\right\}$

Answers (6) $\dfrac{\sqrt{3}}{2}$ (7) not defined

Property - 2(B)

(i) $\sin^{-1}(\sin x) = x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ (ii) $\cos^{-1}(\cos x) = x; \quad 0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x; \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ (iv) $\cot^{-1}(\cot x) = x; \quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x; \quad 0 \le x \le \pi, x \ne \dfrac{\pi}{2}$ (vi) $\text{cosec}^{-1}(\text{cosec } x) = x; \quad x \ne 0, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

These are equal to identity function for a short interval of x only.
(See the graphs on page 19-20)

Solved Example # 4

Find the value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

Solution.

Let $y = \tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

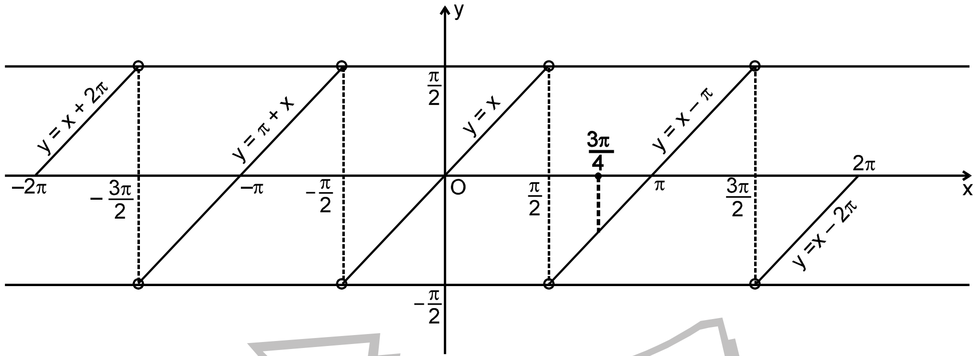
Note $\because$ $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\because \quad \frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$

$\because \quad \frac{3\pi}{4} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\because$ graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\frac{\pi}{2} < x < \frac{3\pi}{2}$,

then $y = \tan^{-1}(\tan x)$ can be written as

$y = x - \pi$

$\therefore \quad y = \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi \quad \therefore \quad$ **$y = -\frac{\pi}{4}$ 123**

solved Example # 5

Find the value of $\sin^{-1}(\sin 7)$

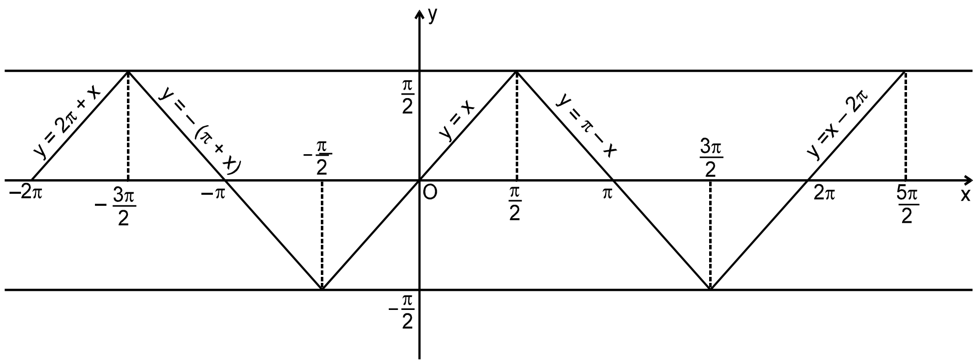
Solution.

Let $y = \sin^{-1}(\sin 7)$

Note : $\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\because \quad 2\pi < 7 < \frac{5\pi}{2}$

$\because$ graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \frac{5\pi}{2}$ then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore$ **$\sin^{-1}(\sin 7) = 7 - 2\pi$**

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because \quad -2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that
$\sin^{-1}(\sin(-5)) = 2\pi + (-5)$
$= 2\pi - 5$

Self practice problems:

(8) Find the value of $\cos^{-1}(\cos 13)$

(9) Find $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$, $\cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$

Ans. (8) $13 - 4\pi$
(9) $\sin^{-1}(\sin\theta) = 3\pi - \theta$; $\cos^{-1}(\cos\theta) = \theta - 2\pi$;
$\tan^{-1}(\tan\theta) = \theta - 3\pi$; $\cot^{-1}(\cot\theta) = \theta - 2\pi$

Property - 2(C)

(i) $\sin^{-1}(-x) = -\sin^{-1}x, \quad -1 \le x \le 1$ (ii) $\tan^{-1}(-x) = -\tan^{-1}x, \quad x \in R$
(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \quad -1 \le x \le 1$ (iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \quad x \in R$

The functions $\sin^{-1}x$, $\tan^{-1}x$ and $\text{cosec}^{-1}x$ are odd functions and rest are neither even nor odd.

Solved Example # 6

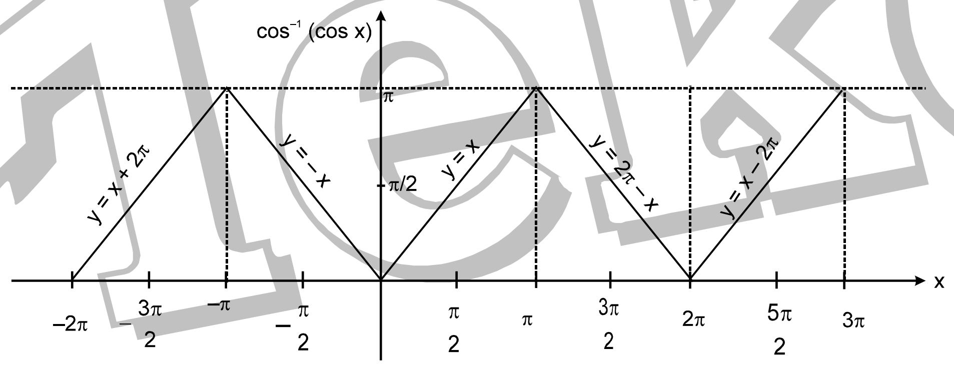
Find the value of $\cos^{-1}\{\sin(-5)\}$

Solution.

Let $y = \cos^{-1}\{\sin(-5)\}$
$= \cos^{-1}(-\sin 5)$ $\quad\because$ **$\cos^{-1}(-x) = \pi - \cos^{-1}x, |x| \le 1$**
$= \pi - \cos^{-1}(\sin 5)$
$= \pi - \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\}$ ....(i)

$\because \quad -2\pi < \left(\frac{\pi}{2} - 5\right) < -\pi$

$\because$ graph of $\cos^{-1}(\cos x)$ is as :



$\because$ from the graph we can see that if $-2\pi \le x \le -\pi$
then $y = \cos^{-1}(\cos x)$ can be written as $y = x + 2\pi$

$\therefore$ from the graph $\cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\} = \left(\frac{\pi}{2} - 5\right) + 2\pi = \left(\frac{5\pi}{2} - 5\right)$

$\therefore$ from equation (i), we get

$\therefore \quad y = \pi - \left(\frac{5\pi}{2} - 5\right)$

$\Rightarrow$ **$y = 5 - \frac{3\pi}{2}$** **Ans.**

Self practice problems:

Find the value of the following

(10) $\cos^{-1}(-\cos 4)$ (11) $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$

(12) $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$

Answers. (10) $4 - \pi$ (11) $\frac{\pi}{8}$ (12) $\left(\frac{1}{4} - \frac{\pi}{2}\right)$

Property - 2(D)

(i) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $x \le -1, x \ge 1$ (ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $x \le -1, x \ge 1$

(iii) $\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; x < 0 \end{cases}$

Solved Example # 7

Find the value of tan $\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Solution

Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(i)

$\because$ **$\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$**

$\therefore$ equation (i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because$ **$\cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$**

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right)$ $\Rightarrow$ $y = -\frac{3}{2}$

Self practice problems:

Find the value of the followings

(13) $\sec\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$ (14) $\text{cosec}\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

Answers. (13) $\frac{3}{2}$ (14) $-\sqrt{3}$

Property - 2(E)

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $-1 \le x \le 1$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, $|x| \ge 1$

Solved Example # 8

Find the value of sin ($2\cos^{-1}x + \sin^{-1}x$) when $x = \frac{1}{5}$

Solution.

Let $y = \sin[2\cos^{-1}x + \sin^{-1}x]$

$\because$ **$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $|x| \le 1$**

$\therefore$ $y = \sin\left[2\cos^{-1}x + \frac{\pi}{2} - \cos^{-1}x\right]$

$= \sin\left[\frac{\pi}{2} + \cos^{-1}x\right]$

$= \cos(\cos^{-1}x)$ $\because$ $x = \frac{1}{5}$

$\therefore$ $y = \cos\left(\cos^{-1}\frac{1}{5}\right)$ ....(i)

$\because$ **$\cos(\cos^{-1}x) = x$ if $x \in [-1, 1]$**

$\because$ $\frac{1}{5} \in [-1, 1]$

$\therefore$ $\cos\left(\cos^{-1}\frac{1}{5}\right) = \frac{1}{5}$ $\therefore$ from equation (i), we get

$\therefore$ $y = \frac{1}{5}$.

Self practice problems:

Solve the following equations

(15) $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$

(16) $4\sin^{-1}x = \pi - \cos^{-1}x$

Answers. (15) $x = 1$ (16) $x = \frac{1}{2}$

Property - 2(F)

(i) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$, $-1 \le x \le 1$

(ii) $\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \frac{1}{x}$, $x \in R$, $x \ne 0$

(iii) $\text{cosec}(\sec^{-1}x) = \sec(\text{cosec}^{-1}x) = \frac{|x|}{\sqrt{x^2-1}}$, $|x| > 1$

Solved Example # 9

Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Solution.

Let $y = \sin\left(\tan^{-1}\frac{3}{4}\right)$ ....(i)

Note : To find y we use $\sin(\sin^{-1}x) = x, -1 \le x \le 1$

For this we convert $\tan^{-1}x$ in $\sin^{-1}x$

Let $\theta = \tan^{-1}\frac{3}{4}$ $\Rightarrow$ $\tan\theta = \frac{3}{4}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ $\sin\theta = \frac{3}{5}$

$\therefore$ $\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{3}{5}\right)$ ....(ii)

$\because$ $\theta \in \left(0, \frac{\pi}{2}\right)$ $\Rightarrow$ $\sin^{-1}(\sin\theta) = \theta$

$\therefore$ equation (ii) can be written as :

$\therefore$ $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ $\because$ $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ $\Rightarrow$ $\tan^{-1}\left(\frac{3}{4}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

$\therefore$ from equation (i), we get $\therefore$ $y = \sin\left(\sin^{-1}\frac{3}{5}\right)$

$y = \frac{3}{5}$

Solved Example # 10

Find the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Solution.

Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ....(i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

∴ equation (i) becomes

$y = \tan\left(\frac{\theta}{2}\right)$ ....(ii)

∵ $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}} = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ....(iii)

∵ $\theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)$

∴ $\tan\frac{\theta}{2} > 0$

∴ from equation (iii), we get

$\tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$

∴ from equation (ii), we get

∴ $\mathbf{y = \left(\frac{3-\sqrt{5}}{2}\right)}$ **Ans.**

Solved Example # 11

Find the value of cos $(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Solution.

Let $y = \cos[2\cos^{-1}x + \sin^{-1}x]$

∵ $\mathbf{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$, $\mathbf{|x| \le 1}$

∴ $y = \cos\left[2\cos^{-1}x + \frac{\pi}{2} - \cos^{-1}x\right]$

$= \cos\left[\frac{\pi}{2} + \cos^{-1}x\right]$

$= -\sin(\cos^{-1}x)$ ∵ $x = \frac{1}{5}$

∴ $y = -\sin\left(\cos^{-1}\frac{1}{5}\right)$ ....(i)

∵ $\mathbf{\sin(\cos^{-1}x) = \sqrt{1-x^2}}$, $\mathbf{|x| \le 1}$

∴ $\sin\left(\cos^{-1}\frac{1}{5}\right) = \sqrt{1-\frac{1}{25}} = \frac{\sqrt{24}}{5}$

∴ from equation (i), we get

$y = -\frac{\sqrt{24}}{5}$

Aliter : Let $\cos^{-1}\frac{1}{5} = \theta \Rightarrow \cos\theta = \frac{1}{5}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$

∴ $\sin\theta = \frac{\sqrt{24}}{5}$

∴ $\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$ ....(ii)

∵ $\theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin^{-1}(\sin\theta) = \theta$

$\therefore$ equation (ii) can be written as

$$\theta = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right) \qquad \because \qquad \theta = \cos^{-1}\left(\frac{1}{5}\right)$$

$$\Rightarrow \qquad \cos^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$$

Now equation (i) can be written as

$$y = -\sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\} \qquad \text{........(iii)}$$

$$\because \quad \frac{\sqrt{24}}{5} \in [-1, 1] \qquad \therefore \qquad \sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\} = \frac{\sqrt{24}}{5}$$

$\therefore$ from equation (iii), we get

$$y = -\frac{\sqrt{24}}{5}$$

Self practice problems:

Find the value of the followings :

(17) $\tan\left(\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$ (18) $\sec\left(\cot^{-1}\frac{16}{63}\right)$

(19) $\sin\left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$ (20) $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$

Answers : (17) $\frac{4}{5}$ (18) $\frac{65}{16}$ (19) $\frac{2\sqrt{5}}{5}$ (20) $\frac{-7}{17}$

3. Identities of Addition and Substraction:

A.

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$, $x \ge 0, y \ge 0$ & $(x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$, $x \ge 0, y \ge 0$ & $x^2 + y^2 > 1$

Note that: $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1}x + \sin^{-1}y \le \frac{\pi}{2}$

$x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$, $x \ge 0, y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $x > 0, y > 0$ & $xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}$, $x > 0, y > 0$ & $xy > 1$

$= \frac{\pi}{2}$, $x > 0, y > 0$ & $xy = 1$

Note that : $xy < 1 \Rightarrow 0 < \tan^{-1}x + \tan^{-1}y < \frac{\pi}{2}$; $xy > 1 \Rightarrow \frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

B.

(i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$, $x \ge 0, y \ge 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$, $x \ge 0, y \ge 0, x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$, $x \ge 0, y \ge 0$

Note: For $x < 0$ and $y < 0$ these identities can be used with the help of properties 2(C) i.e. change x and y to $-x$ and $-y$ which are positive .

Solved Example # 12

Show that $\mathbf{\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\frac{84}{85}}$

Solution.

$\because$ $\dfrac{3}{5} > 0,\ \dfrac{15}{17} > 0$ and $\left(\dfrac{3}{5}\right)^2 + \left(\dfrac{15}{17}\right)^2 = \dfrac{8226}{7225} > 1$

$\therefore$ $\sin^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{15}{17} = \pi - \sin^{-1}\left(\dfrac{3}{5}\sqrt{1-\dfrac{225}{289}} + \dfrac{15}{17}\sqrt{1-\dfrac{9}{25}}\right)$

$= \pi - \sin^{-1}\left(\dfrac{3}{5}\cdot\dfrac{8}{17} + \dfrac{15}{17}\cdot\dfrac{4}{5}\right)$

$= \pi - \sin^{-1}\left(\dfrac{84}{85}\right)$

Solved Example # 13

Evaluate:

$\mathbf{\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}}$

Solution.

Let $z = \cos^{-1}\dfrac{12}{13} + \sin^{-1}\dfrac{4}{5} - \tan^{-1}\dfrac{63}{16}$

$\because$ $\mathbf{\sin^{-1}\frac{4}{5} = \frac{\pi}{2} - \cos^{-1}\frac{4}{5}}$

$\therefore$ $z = \cos^{-1}\dfrac{12}{13} + \left(\dfrac{\pi}{2} - \cos^{-1}\dfrac{4}{5}\right) - \tan^{-1}\dfrac{63}{16}$.

$z = \dfrac{\pi}{2} - \left(\cos^{-1}\dfrac{4}{5} - \cos^{-1}\dfrac{12}{13}\right) - \tan^{-1}\dfrac{63}{16}$ ....(i)

$\because$ $\dfrac{4}{5} > 0,\ \dfrac{12}{13} > 0$ and $\dfrac{4}{5} < \dfrac{12}{13}$

$\therefore$ $\cos^{-1}\dfrac{4}{5} - \cos^{-1}\dfrac{12}{13} = \cos^{-1}\left[\dfrac{4}{5}\times\dfrac{12}{13} + \sqrt{1-\dfrac{16}{25}}\sqrt{1-\dfrac{144}{169}}\right] = \cos^{-1}\left(\dfrac{63}{65}\right)$

$\therefore$ equation (i) can be written as

$z = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{63}{65}\right) - \tan^{-1}\left(\dfrac{63}{16}\right)$

$z = \sin^{-1}\left(\dfrac{63}{65}\right) - \tan^{-1}\left(\dfrac{63}{16}\right)$ ....(ii)

$\because$ $\sin^{-1}\left(\dfrac{63}{65}\right) = \tan^{-1}\left(\dfrac{63}{16}\right)$

$\therefore$ from equation (ii), we get

$\therefore$ $z = \tan^{-1}\left(\dfrac{63}{16}\right) - \tan^{-1}\left(\dfrac{63}{16}\right)$ $\Rightarrow$ **z = 0 Ans.**

Solved Example # 14

Evaluate $\tan^{-1} 9 + \tan^{-1} \frac{5}{4}$

Solution.

$\because \quad 9 > 0, \frac{5}{4} > 0$ and $9\left(\frac{5}{4}\right) > 1$

$\therefore \quad \tan^{-1} 9 + \tan^{-1} \frac{5}{4} = \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9.\frac{5}{4}}\right)$

$= \pi + \tan^{-1}(-1)$

$= \pi - \frac{\pi}{4}.$

$\tan^{-1} 9 + \tan^{-1} \frac{5}{4} = \frac{3\pi}{4}.$

Self practice problems:

(21) Evaluate $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

(22) If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}\lambda$ then find 'λ'

(23) Prove that $2\cos^{-1}\frac{3}{\sqrt{13}} + \cot^{-1}\frac{16}{63} + \frac{1}{2}\cos^{-1}\frac{7}{25} = \pi$

Solve the following equations

(24) $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$ (25) $\sin^{-1}x + \sin^{-1}2x = \frac{2\pi}{3}$

Answers. (21) $\frac{\pi}{2}$ (22) $\lambda = -\frac{19}{9}$ (24) $x = \frac{1}{6}$ (25) $x = \frac{1}{2}$

C.

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$

(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$

(iii) $\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$

(iv) $\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{cases}$

(v) $\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

(See the graphs on page 20)

Solved Example # 15

Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1} x$ and also draw its graph.

Solution.

Let $y = \cos^{-1}(4x^3 - 3x)$

Note $\because$ **Domain : [–1, 1] and range : [0, π]**

Let $\cos^{-1} x = \theta \Rightarrow \theta \in [0, \pi]$ and $x = \cos\theta$

$\therefore \quad y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$y = \cos^{-1}(\cos 3\theta)$ ....(i)

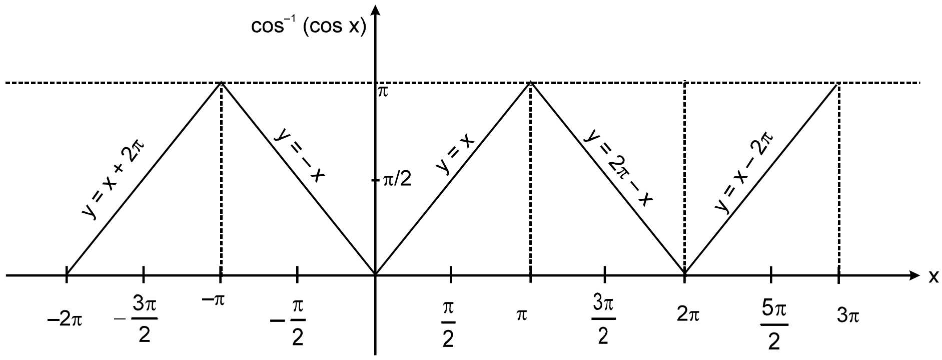


Fig.: Graph of $\cos^{-1}(\cos x)$

$\because \quad \theta \in [0, \pi]$

$\therefore \quad 3\theta \in [0, 3\pi]$

$\therefore$ to define $y = \cos^{-1}(\cos 3\theta)$, we consider the graph of $\cos^{-1}(\cos x)$ in the interval $[0, 3\pi]$. Now, from the above graph we can see that

(i) if $0 \le 3\theta \le \pi \Rightarrow \cos^{-1}(\cos 3\theta) = 3\theta$

$\therefore$ from equation (i), we get

$y = 3\theta$ if $0 \le 3\theta \le \pi$

$\Rightarrow y = 3\theta$ if $0 \le \theta \le \frac{\pi}{3}$

$\Rightarrow$ **$y = 3\cos^{-1}x$ if $\frac{1}{2} \le x \le 1$**

(ii) if $\pi < 3\theta \le 2\pi \Rightarrow \cos^{-1}(\cos 3\theta) = 2\pi - 3\theta$

$\therefore$ from equation (i), we get

$y = 2\pi - 3\theta$ if $\pi < 3\theta \le 2\pi$

$\Rightarrow y = 2\pi - 3\theta$ if $\frac{\pi}{3} < \theta \le \frac{2\pi}{3}$

$y = 2\pi - 3\cos^{-1}x$ if $-\frac{1}{2} \le x < \frac{1}{2}$

(iii) $2\pi < 3\theta \le 3\pi \Rightarrow \cos^{-1}(\cos 3\theta) = -2\pi + 3\theta$

$\therefore$ from equation (i), we get

$\Rightarrow y = -2\pi + 3\theta$ if $2\pi < 3\theta \le 3\pi$

$\Rightarrow y = -2\pi + 3\theta$ if $\frac{2\pi}{3} < \theta \le \pi$

$\Rightarrow$ **$y = -2\pi + 3\cos^{-1}x$ if $-1 \le x < -\frac{1}{2}$**

$\therefore$ from (i), (ii) & (iii), we get

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x & ; \ \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & ; \ -\frac{1}{2} \le x < \frac{1}{2} \\ -2\pi + 3\cos^{-1}x & ; \ -1 \le x < -\frac{1}{2} \end{cases}$$

Graph :

For $y = \cos^{-1}(4x^3 - 3x)$

domain : $[-1, 1]$

range : $[0, \pi]$

(i) if $\frac{1}{2} \le x \le 1$, $y = 3\cos^{-1}x$.

$\Rightarrow \quad \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2}$ ....(i)

$\Rightarrow \quad \frac{dy}{dx} < 0$ if $x \in \left[\frac{1}{2}, 1\right)$

$\Rightarrow$ decreasing if $x \in \left[\frac{1}{2}, 1\right)$

again if we differentiate equation (i) w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = -\frac{3x}{(1-x^2)^{3/2}}$$

$\Rightarrow$ $\frac{d^2y}{dx^2} < 0$ if $x \in \left[\frac{1}{2}, 1\right)$ $\Rightarrow$ concavity downwards if $x \in \left[\frac{1}{2}, 1\right)$

(ii) if $-\frac{1}{2} \le x < \frac{1}{2}$, $y = 2\pi - 3\cos^{-1} x$.

$\therefore$ $\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$ $\Rightarrow$ $\frac{dy}{dx} > 0$ if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow$ increasing if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ and $\frac{d^2y}{dx^2} = \frac{3x}{(1-x^2)^{3/2}}$

(a) if $x \in \left[-\frac{1}{2}, 0\right)$ then $\frac{d^2y}{dx^2} < 0$

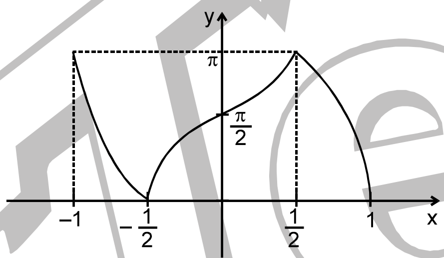
$\Rightarrow$ concavity downwards if $x \in \left[-\frac{1}{2}, 0\right)$

(b) if $x \in \left(0, \frac{1}{2}\right)$ then $\frac{d^2y}{dx^2} > 0$

$\Rightarrow$ concavity upwards if $x \in \left(0, \frac{1}{2}\right)$

(iii) Similarly if $-1 \le x < -\frac{1}{2}$ then $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$.

$\therefore$ the graph of $y = \cos^{-1}(4x^3 - 3x)$ is as



Self practice problems:

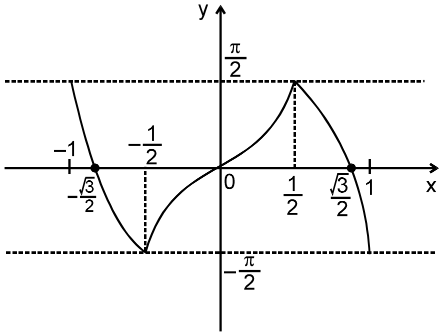
(26) Define $y = \sin^{-1}(3x - 4x^3)$ in terms of $\sin^{-1}x$ and also draw its graph.

(27) Define $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ in terms of $\tan^{-1} x$ and also draw its graph.

Answers

(26) $$y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3\sin^{-1} x & ; \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & ; \quad \frac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1} x & ; \quad -1 \le x < -\frac{1}{2} \end{cases}$$

$\therefore$ graph of $y = \sin^{-1}(3x - 4x^3)$



(27) $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} 3\tan^{-1}x & ; \quad -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & ; \quad -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \quad \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$

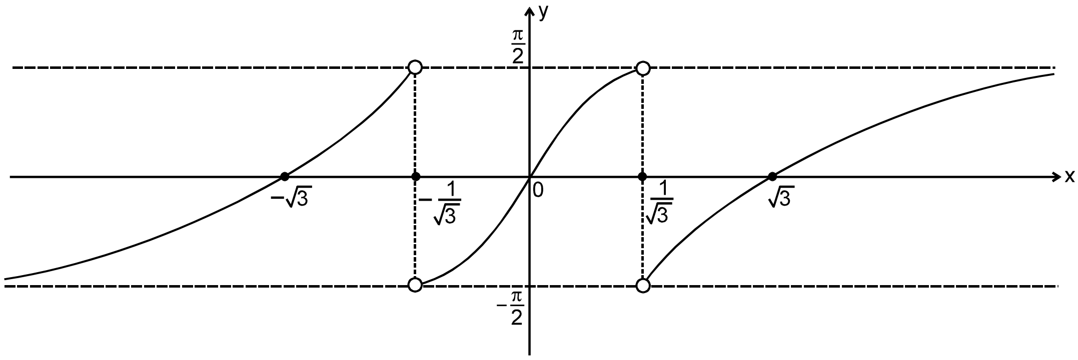


Fig.: Graph of $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

D.

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x + y + z - xyz}{1 - xy - yz - zx}\right]$ if, $x > 0$, $y > 0$, $z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$

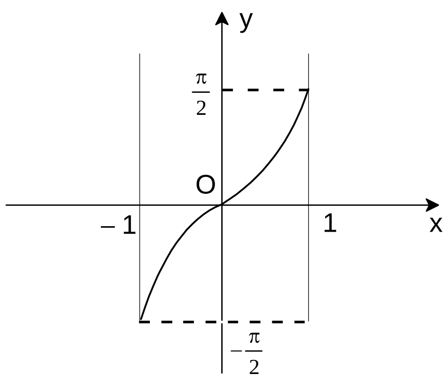
(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(iv) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

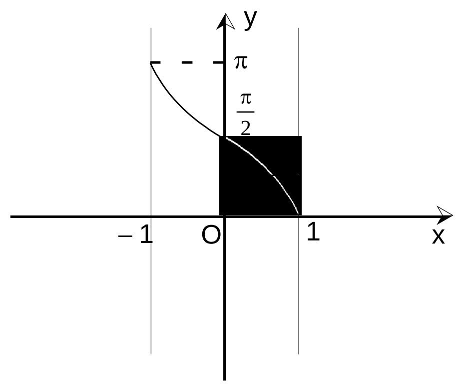
Inverse Trigonometric Functions
Some Useful Graphs

1.

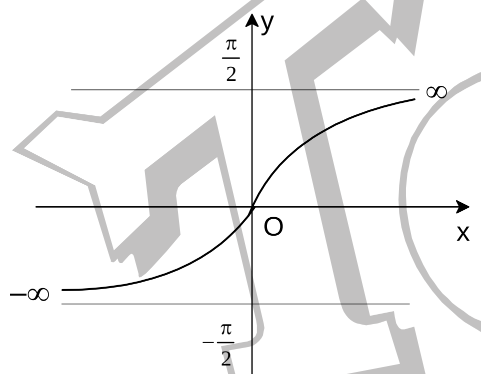
(i) $y = \sin^{-1} x,\ |x| \le 1,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



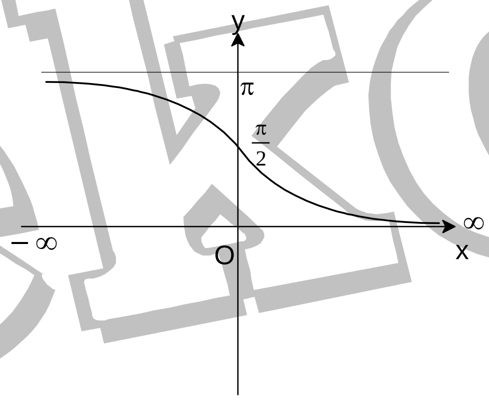
(ii) $y = \cos^{-1} x,\ |x| \le 1,\ y \in [0, \pi]$



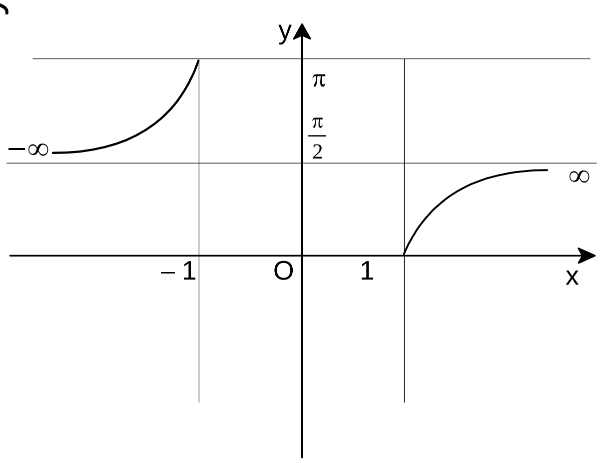
(iii) $y = \tan^{-1} x,\ x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,



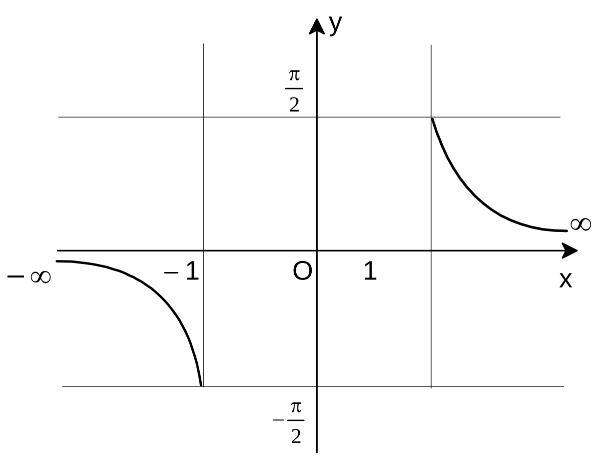
(iv) $y = \cot^{-1} x,\ x \in R,\ y \in (0, \pi)$



(v) $y = \sec^{-1} x,\ |x| \ge 1,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

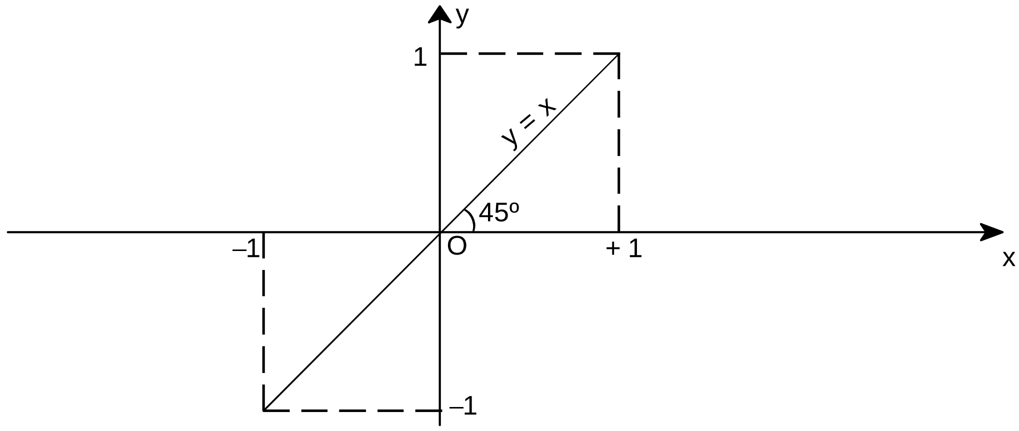


(vi) $y = \text{cosec}^{-1} x,\ |x| \ge 1,\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

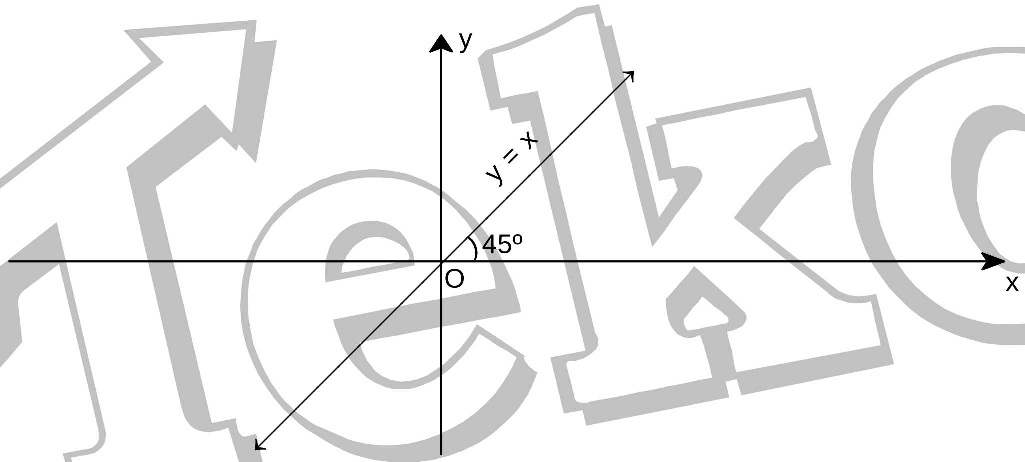


Part - 2(A)

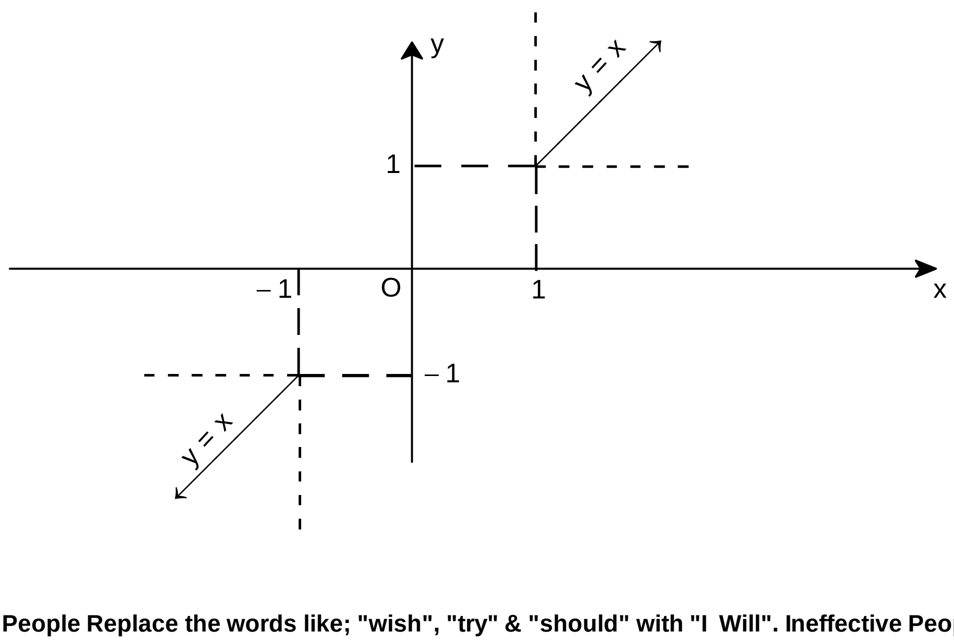
(i) $y = \sin(\sin^{-1}x) = \cos(\cos^{-1}x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$; y is aperiodic



(ii) $y = \tan(\tan^{-1}x) = \cot(\cot^{-1}x) = x$, $x \in R$, $y \in R$; y is aperiodic

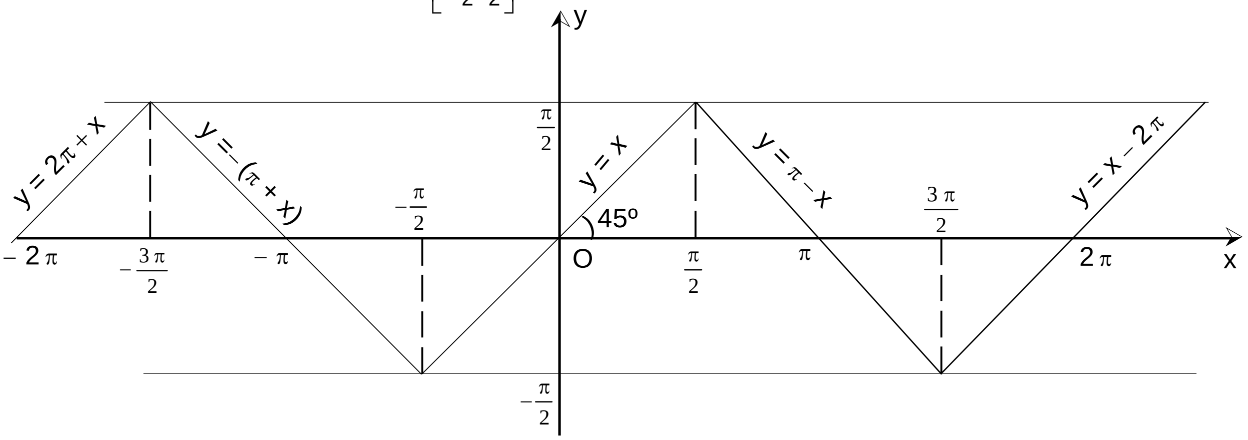


(iii) $y = \text{cosec}(\text{cosec}^{-1}x) = \sec(\sec^{-1}x) = x$, $|x| \geq 1$, $|y| \geq 1$; y is aperiodic

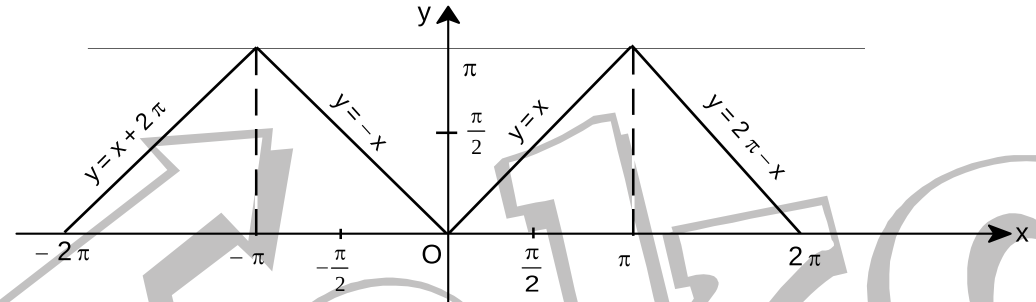


Part -2(B)

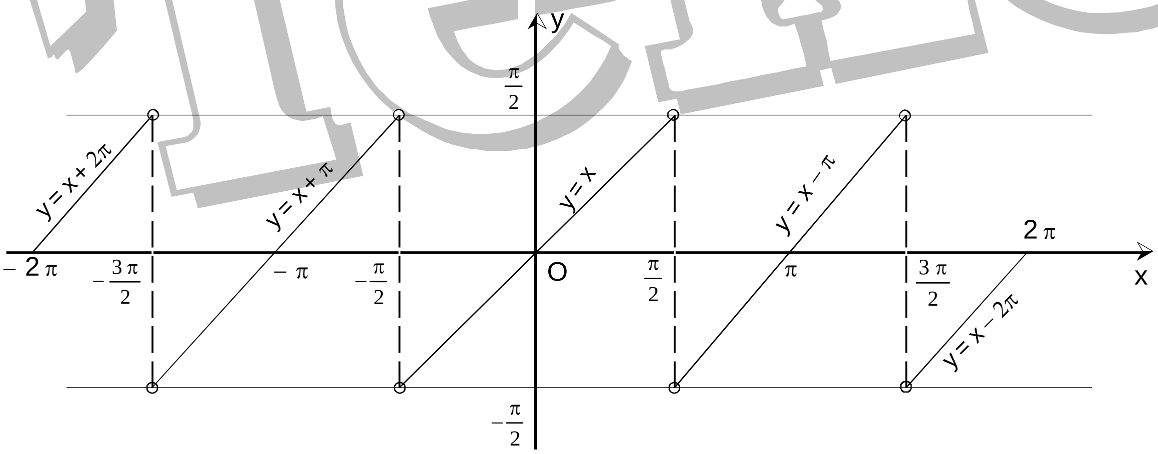
(i) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is periodic with period 2π



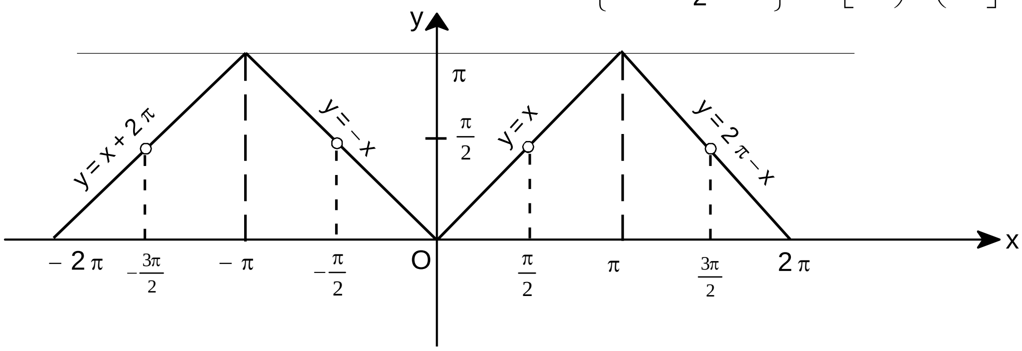
(ii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, is periodic with period 2π



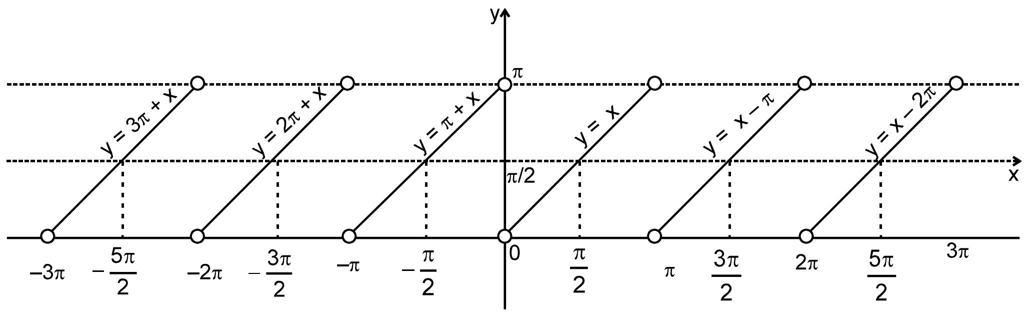
(iii) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{(2n-1)\frac{\pi}{2},\ n \in I\right\}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is periodic with period π



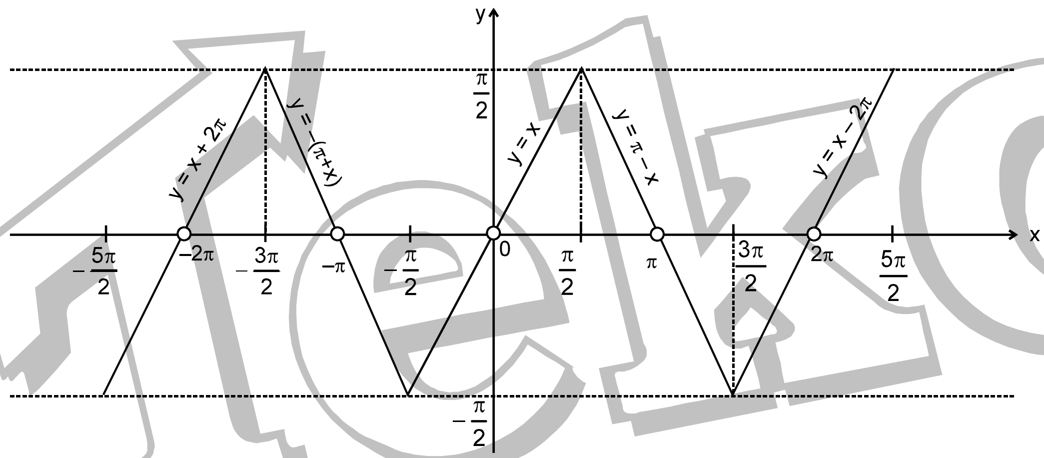
(iv) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π; $x \in R - \left\{(2n-1)\frac{\pi}{2},\ n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(v) $y = \cot^{-1}(\cot x)$, y is periodic with period π; $x \in R - \{n\pi, n \in I\}$, $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right)$

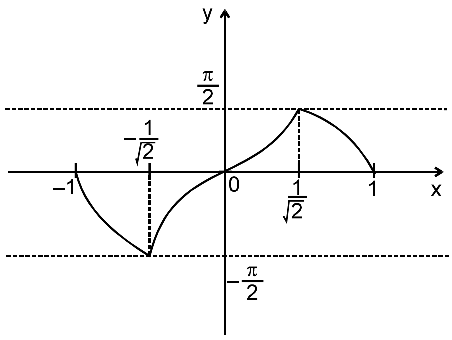


(vi) $y = \text{cosec}^{-1}(\text{cosec}\, x)$, y is periodic with period 2π; $x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

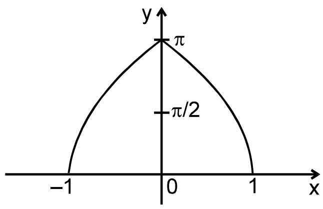


Part - 3(C)

(i) graph of $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

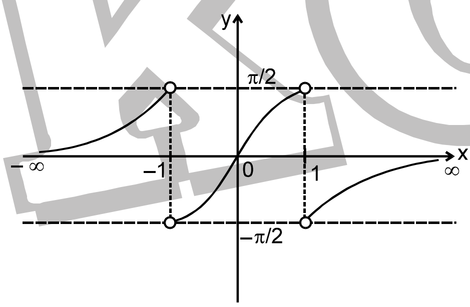


(ii) graph of $y = \cos^{-1}(2x^2 - 1)$

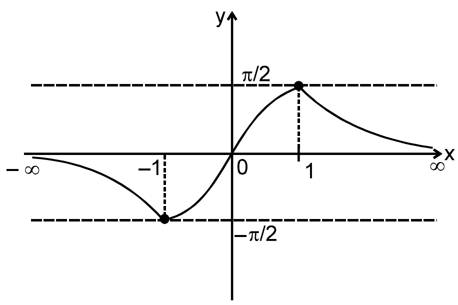


Note : In this graph it is advisable not to check its derivability just by the inspection of the graph because it is difficult to judge from the graph that at x = 0 there is a shapr corner or not.

(iii) graph of $y = \tan^{-1}\frac{2x}{1-x^2}$



(iv) graph of $y = \sin^{-1}\frac{2x}{1+x^2}$



(v) graph of $y = \cos^{-1}\frac{1-x^2}{1+x^2}$